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Small volume expansion of almost supersymmetric large N theories

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ABSTRACT

We consider the small-volume dynamics of nonsupersymmetric orbifold and orientifold field theories defined on a three-torus, in a test of the claimed planar equivalence between these models and appropriate supersymmetric "parent models". We study one-loop effective potentials over the moduli space of flat connections and find that planar equivalence is preserved for suitable averages over the moduli space. On the other hand, strong nonlinear effects produce local violations of planar equivalence at special points of moduli space. In the case of orbifold models, these effects show that the "twisted" sector dominates the low-energy dynamics.

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1. Introduction

Nonsupersymmetric gauge theories with an effectively supersymmetric large-N limit have been the subject of considerable recent interest. The main examples involve the so-called orbifold [1] and orientifold [2] field theories. The prototype of the first class of models is a $U(N)^k$ quiver theory with bi-fundamental fermions and $\mathbf{Z}_k$ global symmetry, whose restriction to planar diagrams is equivalent, up to coupling rescalings, to the analogous planar-diagram approximation of a $U(kN)$ Super Yang–Mills (SYM) model with minimal $\mathcal{N} = 1$ supersymmetry. In the case of orientifold field theories, $SU(N)$ gauge theory with Dirac fermions in two-index representations (either symmetric or antisymmetric) is claimed to be planar-equivalent to the corresponding $\mathcal{N} = 1$, $SU(N)$ gauge theory. In both cases the statement of planar equivalence must be restricted to a particular class of observables that can be appropriately mapped from "parent" to "daughter" theories under the orbifold/orientifold projection. A similar correspondence must be established among the vacua of the theories in question, a very important point given the potentially complicated vacuum structure of these models.

A stronger version of this perturbative correspondence is the hypothesis of nonperturbative planar equivalence (c.f. [3]), namely the parent and daughter theories would have equivalent sectors to leading order in the large-N expansion but non-perturbatively in the 't Hooft coupling, $\lambda = g^2 N$. If true, this stronger version of the planar equivalence would yield interesting predictions for nonperturbative quantities in non-supersymmetric gauge theories, by a modified $1/N$ expansion whose first term is protected by supersymmetry. A remarkable example of this program is the calculation of the chiral condensate of one-flavour nonsupersymmetric QCD (c.f. [4]), in terms of the gaugino condensate of the $\mathcal{N} = 1$ SYM parent gauge theory.

There have been a number of arguments and counter-arguments regarding this important question, especially in the context of the orbifold field theories [5]. In particular, a crucial necessary condition for the strong version of planar equivalence to hold is that the global symmetry group $\mathbf{Z}_k$ should not be spontaneusly broken in either the parent or the daughter vacua under consideration. This restriction leaves $k = 2$ as the only candidate orbifold model to realize the strong version of the planar equivalence conjecture (c.f. [6]), although the calculation of [7] shows that, under compactification on $\mathbf{R}^3 \times \mathbf{S}^1$, the effective three-dimensional theory breaks spontaneously the crucial $\mathbf{Z}_2$ symmetry. On the other hand, the situation is much better for the case of orientifold planar equivalence, where master-field arguments seem to provide a formal proof of the correspondence (c.f. [8]).

In this paper we study the phenomenon of planar equivalence in a small-volume expansion on a three-torus (see [9] for a review). We take spacetime to be $\mathbf{T}^3 \times \mathbf{R}$ with the $\mathbf{R}$ factor representing time, and $\mathbf{T}^3$ a straight torus of size L. In the limit of a small torus, there is a clean Wilsonian separation of scales between "slow" degrees of freedom, i.e. zero modes on $\mathbf{T}^3$, and the non-zero modes, i.e. the "fast" variables in the language of the Born–Oppenheimer approximation. For asymptotically free theories, this limit is accessible to weak-coupling methods, and a systematic effective action over the configuration space of the zero modes can be constructed (c.f. [10,11]). Our purpose is to use these well-developed techniques to get some insight on the question of non-perturbative planar equivalence.

The paper is organized as follows. In section 2 we consider the constraints induced by the hypothesis of planar equivalence on the graded partition function, the natural generalization of the Witten index for these theories. In section 3 we review the details of the Born–Oppenheimer approximation for gauge theories on tori. In section 4 we apply these results to the basic examples of orbifold/orientifold models and use them to test the claim of planar equivalence. Finally, we offer some concluding remarks in section 5.

2. Planar equivalence and the "planar index"

The naturally protected quantity for minimally supersymmetric systems in finite volume is the supersymmetric index, $I = \mathrm{Tr}\,(-1)^F$, [12]. In the case at hand, we shall be interested in the related graded partition function

$$I(\beta) = \mathrm{Tr}\ \mathcal{P}\ (-1)^F\ e^{-\beta H}\ , \tag{2.1}$$

where $\mathcal{P}$ is a projector introducing possible refinements of the index with respect to extra global symmetries of the problem. In principle, a judicious choice of $\mathcal{P}$ might be necessary to establish the planar equivalence, but we shall suppress it for the time being. Unlike the analogous object in the parent $\mathcal{N}=1$ theory, $I(\beta)$ is not independent of β/L or the dimensionless couplings in the Lagrangian. The interesting question is whether we can establish an approximate BPS character of $I(\beta)$.

The strongest possible statement of planar equivalence would have $I(\beta)$ behaving as a supersymmetric index of a rank N supersymmetric gauge theory, at least in some dynamical limit. This would mean a large-N scaling of order

$$I(\beta) = O(N)\ . \tag{2.2}$$

On the other hand, perturbative planar equivalence poses much weaker constraints on the graded partition function, i.e. it only requires

$$\log\; I(\beta) = O(N)\;, \tag{2.3}$$

since the leading $O(N^2)$ term is a sum of planar diagrams and should vanish as in the supersymmetric parent. Notice that condition (2.3) is exponentially weaker than (2.2). The physical interpretation of the minimal planar equivalence condition (2.3) depends to a large extent on the dynamical regime that we consider. For example, the implications for the structure of the spectrum depend on the value of the ratio β/L. Let us write (2.1) in terms of the spectrum of energy eigenvalues,

$$I(\beta) = \sum_E \;(\Omega_B(E) - \Omega_F(E))\; e^{-\beta\, E}\;, \tag{2.4}$$

with $\Omega_{B,F}$ the boson and fermion density of states. The thermal partition function is given by

$$Z\,(\beta) = \sum_E \;(\Omega_B(E) + \Omega_F(E))\; e^{-\beta\, E}\,. \tag{2.5}$$

In the large-volume limit, $\beta/L \ll 1$, these quantities are dominated by the high-energy asymptotics of the spectrum. In particular, for asymptotically free theories, we expect that the free energy can be approximated by that of a plasma phase. Thus, on dimensional grounds,

$$\log\; Z\,(\beta) = -N^2\, f(\lambda)\, (L/\beta\,)^3 + O(N)\;,$$

up to a vacuum-energy contribution linear in β, with $f(\lambda)$ a function of the 't Hooft coupling $\lambda = g^2 N$, conveniently renormalized at the scale $1/\beta$. From this expression we infer that the asymptotic high-energy behaviour of the density of states is given by

$$\log\; \Omega_{B,F}(E) = \sqrt{N}\, s_{B,F}(\lambda)\, (E\, L)^{3/4} + O(N)\,.$$

Notice that the leading term is of $O(N^2)$ for $E = O(N^2)$. Then, the prediction of planar equivalence boils down to $s_B = s_F$, namely the leading $O(N^2)$ high-energy asymptotics of the density of states is effectively supersymmetric. [1]

Conversely, in the opposite limit $L/\beta \ll 1$ the graded partition function $I(\beta)$ is dominated by low-lying states. In the classical approximation, the Hamiltonian of gauge

[1] This property is very similar to the so-called "misaligned supersymmetry", studied in [13], that characterizes non-supersymmetric string theories without classical tachyonic instabilities.

theories on $\mathbf{T}^3$ has states of vanishing potential energy, corresponding to flat connections on the gauge sector and zero eigenvalues of the Dirac operator on the fermion sector. This implies that there is a neat separation of scales, of order $1/L$, between zero modes and the rest of the degrees of freedom in the limit of small volume. The graded partition function may be written as

$$I(\beta) = \mathrm{Tr}_{\,\mathrm{slow}}\, (-1)^F\, e^{-\beta H_{\mathrm{eff}}}\,, \tag{2.6}$$

with H_{eff} a Wilsonian effective Hamiltonian for the zero modes or "slow variables". For asymptotically free theories, this Hamiltonian may be estimated in a weak-coupling expansion in the small running coupling g^2N, defined at the scale $1/L$ (c.f. [10,11]).

A characteristic behaviour of the supersymmetric index is that the effective Hamiltonian acting on the space of zero modes with energies much smaller than $1/L$ is free, i.e. the Wilsonian effective potential induced by integrating out the non-zero modes vanishes exactly as the result of boson/fermion cancellation. In the non-supersymmetry case, a similar behaviour of (2.6) would require that the effective potential be at most of order $V_{\mathrm{eff}} = O(1/N)$, i.e. it would have to vanish in the large N limit. On the other hand, planar equivalence would require the weaker condition that this effective potential vanishes to leading $O(N^2)$ order or, equivalently $V_{\mathrm{eff}} = O(N)$.

Therefore, the existence of a sort of "planar index" seems to be a stronger property than "minimal" planar equivalence, in the sense discussed in the Introduction. Understanding this fact in detail is important, given the close relationship between the supersymmetric index in finite volume and the gaugino condensates in the standard $\mathcal{N} = 1$ SYM lore [14].

In this paper we examine the zero-mode effective potentials for particular examples of theories in which a form of planar equivalence is expected to hold. To be more precise, we consider the basic $\mathbf{Z}_2$ orbifold model, with gauge group $SU(N) \times SU(N)$ and a Dirac bi-fundamental fermion. We also consider the orientifold model, an $SU(N)$ gauge theory with a Dirac fermion in either the symmetric or antisymmetric two-index representations.

3. Effective potentials for flat connections

We focus on gauge theories with simply-connected gauge group on the torus, namely, we can take periodic boundary conditions for all the fields involved, generating the maximal possible set of zero modes. The configuration space of bosonic zero modes is the moduli space of flat connections on $\mathbf{T}^3$, which in turn can be characterized by a commuting triple of holonomies, modulo gauge transformations (c.f. [9]). If the gauge group is taken to be

a direct product of $SU(N)$ factors, the moduli space has the same product structure, with each factor given by direct product of Cartan tori $\mathbf{T}_{\mathrm{C}}^{N-1}$ for each holonomy, modulo the Weyl group $W = S_N$, which acts by permutations of the holonomies' eigenvalue spectrum,[2]

$$\mathcal{M}_{SU(N)} = \frac{\left(\mathbf{T}_{\mathrm{C}}{}^{N-1}\right)^3}{W} \ .$$

As coordinates on $\mathcal{M}$, we choose constant dimensionless gauge fields in the Cartan subalgebra $L\,\vec{A} = \sum_{a=1}^{N-1} \vec{C}^a\, H^a$. We refer to the torus $(\mathbf{T}_{\mathrm{C}}{}^{N-1})^3$ as the "toron valley", defined as the product of three copies of $\mathbf{R}^{N-1}/2\pi\widetilde{\Lambda}_r$, where $\widetilde{\Lambda}_r$ is the dual of the root lattice, i.e. we identify $\vec{C}$ modulo translations by $4\pi\alpha\,\mathbf{Z}^3$, with α a root and $\mathbf{Z}^3$ a three-vector of integers. The Weyl group acts by reflections on the hyperplanes orthogonal to the roots, and makes $\mathcal{M}$ into an orbifold.

Fermionic zero modes are given by solutions of $\not{D}\psi = 0$. At a generic point on $\mathcal{M}$, flatness of the gauge field implies that ψ is actually covariantly constant, i.e. invariant under the holonomies that parametrize $\mathcal{M}$. If ψ is in the adjoint representation of the gauge group, this puts the fermion zero modes on the Cartan subalgebra for generic points on $\mathcal{M}$. However, for either fundamental or symmetric/antisymmetric representations, fermion zero modes will be supported on submanifolds of $\mathcal{M}$ of zero measure. For this reason, when discussing the non-supersymmetric theories, we shall consider only the bosonic zero modes as explicit variables and integrate out all the fermionic degrees of freedom.

Under these circumstances, the effective Lagrangian around a *generic* point of the bosonic moduli space takes the form

$$\mathcal{L}_{\mathrm{eff}} = \frac{L}{2g^2}\,\mathrm{Tr}\,\left(\partial_t\,\vec{C}\right)^2 - V_{\mathrm{eff}}(\,\vec{C}\,)\,, \tag{3.1}$$

where the effective potential is the sum of bosonic and fermionic parts. In the one-loop approximation and in background-field gauge it is given by

$$\int_{-\infty}^{\infty} dt\; V_{\mathrm{eff}}(\vec{C}\,) = \mathrm{Tr}_{\,\mathrm{Ad}}\,\log\left(-D^2_{\vec{C}}\right) - \sum_R \mathrm{Tr}_{\,R}\,\log\left(\,i\not{D}_{\vec{C}}\,\right)\;, \tag{3.2}$$

where R stands for the fermion representations. The operator $-D^2$ acts on scalar functions on $\mathbf{R}\times\mathbf{T}^3$, after we have taken into account the contribution from gauge-field polarizations

[2] In what follows, we focus on gauge groups with simple $SU(N)$ factors, neglecting for example the $U(1)$ factors that arise in orbifold models, which would only contribute subleading effects in the large N limit.

and ghosts. The normalization convention for $i\not{D}$ regards fermion representations as carried by Majorana or Weyl spinors, so that representations carried by Dirac spinors actually involve $R \oplus \bar{R}$, contributing with an extra factor of 2 to the total potential.

Using the fact that $(i\not{D})^2 = -D^2$ for flat connections, we simply need to compute the determinant of the operator $-D^2$ in different representations. Since we work on the space of constant abelian connections, the traces in (3.2) involve a sum over the relevant weight spaces and we can write (c.f. [10,11,9])

$$V_{\rm eff}(\vec{C}\,) = \sum_{\alpha} V\left(\alpha \cdot \vec{C}\right) - \sum_{R} \sum_{\mu \in R} V\left(\mu \cdot \vec{C}\right) \,, \tag{3.3}$$

where α runs over the roots of $SU(N)$, μ runs over the weights of the fermion representations R and the dot product refers to the Cartan inner product, i.e. $\alpha \cdot \vec{C} \equiv \sum_{a=1}^{N-1} \alpha_a \vec{C}_a$. In this general expression, the function V is the subtracted scalar determinant

$$\int_{-\infty}^{\infty} dt\, V(\vec{\xi}\,) = {\rm Tr}\ \log\left[-\partial_t^2 - \left(\vec{\partial} + L^{-1}\,\vec{\xi}\right)^2\right]_{\mathbf{R}\times\mathbf{T}^3} - {\rm Tr}\ \log\left[-\partial^2\right]_{\mathbf{R}\times\mathbf{T}^3} \,,$$

which can be computed by standard methods (c.f. [10,15]) with the result

$$V(\vec{\xi}\,) = \frac{2}{\pi^2 L} \sum_{\vec{n}\neq 0} \frac{\sin^2\left(\frac{1}{2}\,\vec{n}\cdot\vec{\xi}\right)}{(\vec{n}\cdot\vec{n}\,)^2} \,, \tag{3.4}$$

up to numerical, $\vec{\xi}$-independent constants. We shall refer to (3.4) as the $SU(2)$ potential, since it coincides with the contribution, up to a factor of 2, of the $SU(2)$ adjoint representation.

These expressions make it clear that the parent supersymmetric theories have vanishing effective potential over $\mathcal{M}$, since fermions are in the adjoint representation and the bosonic and fermionic potentials cancel out one another.

3.1. General properties of the effective potential

The elementary building block of (3.3), the $SU(2)$ potential, has periodicity $\vec{\xi} \rightarrow \vec{\xi} + 2\pi\vec{k}$, $\vec{k} \in \mathbf{Z}^3$ and is symmetric under reflections $\vec{\xi} \rightarrow -\vec{\xi}$. The minima of the positive function $V(\vec{\xi}\,)$ are located at $\vec{\xi}_{\rm min} = 0$ modulo $2\pi\mathbf{Z}^3$, and correspond to conical singularities where $V(\vec{\xi}\,) \sim |\vec{\xi} - \vec{\xi}_{\rm min}|$. Maxima are smooth and sit at $\vec{\xi}_{\rm max} = (\pi, \pi, \pi)$ modulo $2\pi\mathbf{Z}^3$ (c.f. Fig 1).

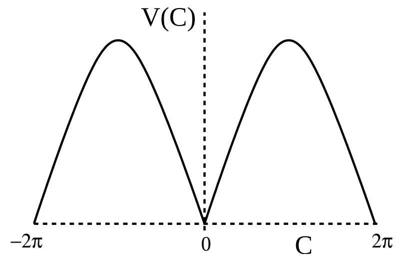

Fig. 1: *$SU(2)$ bosonic potential along a Cartan direction. Conical minima are located at $C = 0 \bmod 2\pi$, smooth maxima at $C = \pi \bmod 2\pi$.*

These properties propagate to the full $SU(N)$ potential, with due consideration of the global group-theory structure for each contributing representation. The potential induced by the pure-gauge degrees of freedom enjoys a translational symmetry under $\vec{C} \to \vec{C} + 4\pi\vec{k}\,\nu^i$ with ν^i any of the N weights of the defining representation. In particular, all zeros of the adjoint potential can be obtained from $\vec{C} = 0$ by the action of this translational symmetry. The induced periodicity on the moduli space is finer than the minimal one, imposed by the toroidal nature of the potential valleys. This periodicity is associated to the action of the global symmetry group $(\mathbf{Z}_N)^3$ of central conjugations, whose characters label the non-abelian electric flux sectors [16].

In general, this symmetry of central conjugations is broken by the fermionic representations down to a subgroup of order $n(R)^3$, where $n(R)$ is the so-called "N-ality" of the representation. It can be defined by the number of Young tableaux, modulo N, or in the highest-weight parametrization

$$\mu_R^{\rm hw} = \sum_{k=1}^{N-1} q_k\, \mu^k \ ,$$

with $\mu^k = \sum_{i=1}^k \nu^i$ the fundamental weights and $q_k \geq 0$, the "N-ality" is given by $n(R) = \sum_k q_k k \bmod N$. Hence, for fermion representations with trivial center, such as the defining "vector" representation, the translations by multiples of $4\pi\nu^i$ do not define a symmetry of the full effective potential.

In general, representations with trivial N-ality will induce potentials that fit within the frame of the bosonic potential, and partial cancellations of some bosonic minima are possible. Representations with non-trivial N-ality will generate potentials that fit better within the frame of the fundamental (vector) representation. The local bosonic mimima will generically survive as global minima, except perhaps the one at the origin.

The contribution of a representation R to the potential is very constrained geometrically by the action of the Weyl group, generated by permutations of the weights. The

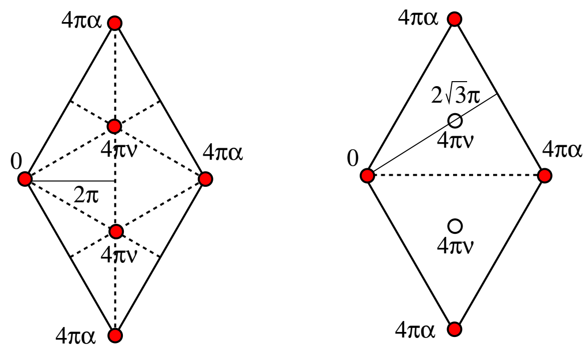

Fig. 2: *The structure of the $SU(3)$ potential for the adjoint representation (on the left) and the defining vector representation (on the right). The Weyl hyperplanes are indicated by dashed lines. We see that the effective Weyl chamber for the vector representation is bigger than the adjoint one by a factor $\sqrt{3}$. The conical points (global minima or maxima) lie on the intersection of Weyl hyperplanes. For the vector representation, we also mark by white circles the smooth critical points of the potential.*

combination of the Weyl reflections and the periodicity properties of the potential single out the $(N-2)$-dimensional hyperplanes given by the linear equations

$$\mu \cdot \vec{C} = 0 \mod 2\pi\, \mathbf{Z}^3 \,.$$

for each weight, there is one such Weyl hyperplane passing through the origin, and all others are parallel and shifted by integer multiples of the normal vector $2\pi\mu/|\mu|^2$. Weyl hyperplanes are local minima of the bosonic potential along the transverse directions (and local maxima of the fermionic potential). Hence, local minima (maxima) of the bosonic (fermionic) potential are located at intersections between Weyl hyperplanes. The fundamental region of the Weyl reflection group is the Weyl chamber, and is enclosed by such Weyl hyperplanes. As a result local minima (maxima) of the bosonic (fermionic) potential are located at the vertices of the Weyl chambers. For a given representation, the induced potential shows structure up to the scale given by the size of the associated Weyl chamber,

$$\ell_R = \frac{2\pi}{|\mu_R|}$$

with μ_R the highest weight of the representation R.

As it stands, the effective potential (3.3) is not smooth at the orbifold singularities of $\mathcal{M}$. The vertices of the Weyl chamber signal either bosonic minima or fermionic maxima. Locally, the behaviour of the potential at these points is conical $V_{\text{eff}} \sim |\vec{C}\,|$, a result of the appearance of extra localized light modes. Focusing on the minima, the leading

non-derivative potential away from the toron valley is given by the commutator term Tr $[C_\mu, C_\nu]^2/g^2$, which induces a mass for generic excitations orthogonal to the toron valley. Zero-point fluctuations of these modes are at the origin of the effective potential considered above. However, near $\vec{C}=0$ (or any of the other orbifold singularities), the N^2-1 constant modes are effectively massless. Upon rescaling $\vec{C} \rightarrow g^{2/3}\vec{C}$, we obtain a homogenous scaling of the effective Hamiltonian yielding a spectrum of order $g^{2/3}/L \sim 1/LN^{1/3}$, with wave functions localized over a region of size $O(g^{2/3}) \sim O(1/N^{1/3})$ around the orbifold points [10]. Away from this "stadium", the effective potential produces a barrier that supresses the wave functions exponentially at large N. In general, barriers are very efficient in supressing tunneling at large N, since the flat connections have a large effective mass of order $m_{\rm eff} \sim LN/\lambda$ at weak coupling, as can be seen from (3.1).

These dynamical considerations show that, unless the cancellation between bosonic and fermionic potentials is accurate up to corrections of $O(1/N)$, the low-lying wave functions will tend to remain rather localized in the large N limit, a situation that is altogether very different from the supersymmetric case, where zero-mode wave functions can be considered uniformly distributed over the moduli space (in fact, constant) despite the existence of orbifold singularities (see [17,12] for further discussion of the subtleties involved).

4. Effective potentials and planar equivalence

At the level of the effective potentials considered in this paper, the property of planar equivalence manifests itself in the cancellation of bosonic and fermionic contributions to leading order in the large-N expansion. Namely, one should find $V_{\rm eff}(\vec{C}\,) = O(N)$ instead of the more generic $O(N^2)$ scaling. Alternatively, one relates the given model to its supersymmetric parent. In many cases, this comparison is rather subtle. For example, in orbifold models the detailed global structure of the toron valleys changes from partent to daughter, since the Weyl groups are clearly different. Hence, the domains of definition of $V_{\rm eff}(\vec{C}\,)$ differ between parent and daughter theories and a precise comparison would involve a further refinement of the projector that appears in (2.1).

For models with the same configuration space, such as the orientifold field theory, the main property that ensures planar equivalence is the equality, to leading $O(N^2)$ order, of the fermion effective actions between parent and daughter, i.e.

$$\mathrm{Tr}_{\,\mathrm{Ad}} \,\log\,(\,i\not{D}\,) - \mathrm{Tr}_{\,\mathrm{R}}\,\log\,(\,i\not{D}\,) = O(N)\;. \tag{4.1}$$

In principle, this property may be established in a strong sense, for any sufficiently smooth gauge connection A, provided it belongs to the configuration space of both theories. Within the weak-field expansion, the perturbative version of planar equivalence ensures (4.1) to any finite order in the one-loop Feynman diagram expansion in the background field. To see this, consider the one-loop diagram with n external legs, proportional to $\mathrm{Tr}_{\,\mathrm{R}} \left(\not{A}/\not{\partial}\right)^n$. It contains a group-theory factor proportional to $\mathrm{Tr}_{\,\mathrm{R}}\, T^{a_1} \dots T^{a_n}$. Reducing this trace to a combination of symmetrized traces, we end up with terms of the form

$$\mathrm{STr}_{\mathrm{R}}\, T^{a_1} \dots T^{a_n} = I_n(R)\, d^{\,a_1 \cdots a_n} + \text{lower order products},$$

where $I_n(R)$ denotes the Dynkin index of the representation R, and the symmetric polynomial $d^{\,a_1 \cdots a_n}$ is the symmetrized trace in the fundamental representation.

Then, the Dynkin index for the symmetric and antisymmetric representations, $S_\pm$, is given by $I_n(S_\pm) = 2(N \pm 2^n)$ (c.f. [18]), to be compared with $I_n(\mathrm{Ad}) = 2N$ for the adjoint representation. The leading Dynkin index is indeed the same in the large-N limit, so that a given diagram with an arbitrary (but *fixed*) number of legs yields the same contribution in the large-N limit in the parent and daughter theories. So far this is just another way of looking at the statement of perturbative planar equivalence. However, $I_n(S_\pm)$ has a subleading (non-planar) term that blows-up exponentially with the number of legs of the diagram. Hence, planar equivalence stated diagram by diagram no longer guarantees that the full non-perturbative effective action will satisfy the planar-equivalence property (4.1). In principle, there could be a non-commutativity between the large-N limit and the non-linearities beyond the weak-field expansion, and this question could be a dynamical one, i.e. depending on the particular gauge connection considered as background.

These considerations show that the behaviour of the exact one-loop potential (3.3) under planar equivalence is a rather non-trivial issue. The purely perturbative version of the planar equivalence is strictly related to the Taylor expansion of the potential around the origin of moduli space $\vec{C} = 0$, and the behaviour at finite distance away from the origin, $|\vec{C}| \sim 1$, remains an open question. In the following subsections we consider in more detail the effective potential in the two main examples of planar equivalence.

4.1. Orientifold effective potential

The orientifold model defined by a $SU(N)$ gauge theory with a Dirac fermion in the antisymmetric representation has an effective potential for the flat connections that we may conveniently write as a superposition of the basic $SU(2)$ potentials (3.4), following the general structure of (3.3).

It is convenient to parametrize the weights and roots of the relevant representations in terms of the N weights, ν^i, of the defining vector representation of $SU(N)$, that satisfy $\nu^i \cdot \nu^j = (N\delta^{ij} - 1)/2N$, so that $\nu^i + \nu^j$ with $i < j$ run over the weights of the antisymmetric two-index representation, whereas $\nu^i - \nu^j$ run over the roots of $SU(N)$ as $i, j = 1, \dots, N$, except for the fact that in this way we count one extra vanishing root, which gives no contribution since $V(0) = 0$ in our additive convention for the vacuum energy. [3] The final form of the potential is

$$V_{\rm ori}(\vec{C}) = \sum_{i,j} V\left((\nu^i - \nu^j)\cdot \vec{C}\right) - 2\sum_{i<j} V\left((\nu^i + \nu^j)\cdot \vec{C}\right) \;, \tag{4.2}$$

with an extra term

$$-2\sum_i V\left(2\nu^i \cdot \vec{C}\right)$$

to be added for the symmetric two-index representation. The first two terms are nominally of $O(N^2)$ from the multiplicity of the index sums, whereas the last term is at most of $O(N)$, and thus may be ignored when discussing the property of (minimal) planar equivalence in this model.

The first nontrivial property of the orientifold potential is the insatbility of the origin of moduli space, $\vec{C} = 0$. To see this, we can explore the potential along the direction of a fundamental weight, say $\vec{C} = \nu^1 \vec{c}$, with very small $\vec{c}$, so that the $SU(2)$ potential can be approximated by $V(\vec{\xi}) \simeq v|\vec{\xi}|$ with v a positive constant.

Evaluating the potential in this direction one obtains

$$V_{\rm ori}(\nu^1 \vec{c}) \approx -N|\vec{c}|$$

up to subleading corrections at large N. Hence, we conclude that the zero-holonomy point at the origin of moduli space becomes severely unstable at large N.

The scale over which the bosonic and fermionic contributions vary is roughly the same, since the size of the Weyl chamber for the two-index representations is

$$\ell_{S_\pm} = \frac{2\pi}{|\nu^i + \nu^j|} = 2\pi + O(1/N), \quad i \neq j\,,$$

whereas the size of the standard Weyl chamber is $\ell_{\rm Ad} = 2\pi/|\nu^i - \nu^j| = 2\pi$. On the other hand, the alignment of the bosonic and fermionic potentials is not perfect. We can see this

[3] The $\vec{C}$-independent part of $V_{\rm eff}$ vanishes to $O(N^2)$ in all the models studied in this paper.

by checking the height of the local conical minima [4], that are inherited from the absolute zeros of the bosonic potential. Conical minima of the bosonic potential are determined by the equations

$$(\nu^i - \nu^j) \cdot \vec{C}^{(0)} = 0 \text{ mod } 2\pi \mathbf{Z}^3$$

for all i, j, which have as the general solution the integer lattice generated by the vectors $4\pi\nu^i$. We can now evaluate the fermionic contribution

$$V_{\mathrm{F}}(\vec{C}^{(0)}) = -2 \sum_{i<j} V\left((\nu^i + \nu^j) \cdot \vec{C}^{(0)} \right)$$

at a generic zero of the bosonic contribution, given by $\vec{C}^{(0)} = 4\pi \sum_i \vec{n}_i \, \nu^i$, with $\vec{n}_i \in \mathbf{Z}^3$. Using the properties of the basic weights we have

$$(\nu^i + \nu^j) \cdot 4\pi \sum_k \vec{n}_k \, \nu^k = 2\pi(\vec{n}_i + \vec{n}_j) - \frac{4\pi}{N} \sum_k \vec{n}_k$$

and, by the periodicity properties of the $SU(2)$ potential, we can write

$$V_{\mathrm{F}}(\vec{C}^{(0)}) \sim -N^2 V \left(\frac{4\pi}{N} \sum_k \vec{n}_k \right) \, .$$

Hence, as we move around the lattice of conical minima from points with $\sum_k \vec{n}_k = O(1)$ to points with $\sum_k \vec{n}_k = O(N)$, the potential scans on a band of energies ranging from $O(N)$ up to $O(N^2)$, with a typical spacing of $O(N)$. As an example of a set of local minima with $O(N^2)$ negative energy, we can consider points in the lattice satisfying $\sum_k \vec{n}_k = (N/4, 0, 0)$ at large values of N such that $N/4$ is an integer. On these points, the fermionic contribution is maximal and given by

$$V_F = -N^2 \, V(\pi) + O(N) \, .$$

In an analogous fashion, the conical maxima of the fermionic potential should be lifted up by the smooth portions of the bosonic potential giving us an intrincate *landscape* of very narrow valleys and peaks on the $O(N)$ scale of relative heights, with some walls rising up to $O(N^2)$ energies. The conclusion is that nonlinear effects at finite distance away from the origin of moduli space tend to spoil the property of planar equivalence, at least when looking at the potentials with high enough resolution.

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[4] The conical singularities are smoothed out when considering the effects of the light non-abelian degrees of freedom, as explained at the end of Section 3.

In fact, it turns out that planar equivalence is still maintained on the average, since the integral of (4.2) over the toron valley is only of $O(N)$. To see this, we can expand the flat connections in the basis of the simple roots $\alpha_s^i = \nu^i - \nu^{i+1}, i = 1, \ldots, N-1$, i.e. we write

$$\vec{C} = \sum_{l=1}^{N-1} \vec{c}_l \; \alpha_s^l$$

and the orientifold potential takes the form

$$V_{\rm ori}(\vec{C}\,) = \sum_{i,j} \left[V\left(\tfrac{1}{2}(\vec{c}_i - \vec{c}_{i-1} - \vec{c}_j + \vec{c}_{j-1})\right) - V\left(\tfrac{1}{2}(\vec{c}_i - \vec{c}_{i-1} + \vec{c}_j - \vec{c}_{j-1})\right)\right] + O(N)\;. \tag{4.3}$$

In this expression, we have neglected terms of $O(N)$ coming from restrictions on the range of the indices. When averaging over the toron valley, the coordinates $\vec{c}_j$ become dummy integration variables, and an appropriate change of variables shows that the bosonic and fermionic terms cancel out upon integration.

We can use similar arguments to show that the average of the squared potential $(V_{\rm ori})^2$ is at most of $O(N^3)$. This shows that the average cancellation of $O(N^2)$ terms in (4.3) does not come from large "plateaus" of $O(N^2)$ energy occupying different $O(1)$ fractions of moduli-space's volume. If that would have been the case, all these smooth regions would contribute to $(V_{\rm eff})^2$ as positive plateaus of $O(N^4)$ energy, producing an average squared potential of $O(N^4)$.

The vanishing of the leading $O(N^4)$ terms in the averaged squared potential suggests that the valleys and peaks of the orientifold potential are localized over small volumes of the moduli space in the large N limit. Still, the wave functions will show strong localization properties and our analysis sheds no new light on the important question of chiral symmetry breaking in these models (non-trival expectation values of fermion bilinears in the infinite volume limit).

4.2. Orbifold effective potential

For the basic orbifold model with $SU(N) \times SU(N)$ group and a bi-fundamental Dirac fermion we have a potential defined over the direct product of toron valleys parametrized by $\vec{C}_1$ and $\vec{C}_2$ flat connections. It has the form

$$V_{\rm orb}(\vec{C}_1, \vec{C}_2) = \sum_{i,j} V\left((\nu^i - \nu^j\,)\cdot \vec{C}_1\right) + \sum_{i,j} V\left((\nu^i - \nu^j\,)\cdot \vec{C}_2\right) - 2\sum_{i,j} V\left(\nu^i \cdot \vec{C}_1 - \nu^j \cdot \vec{C}_2\right). \tag{4.4}$$

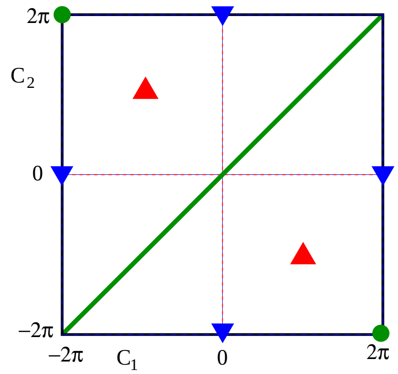

Fig. 3: *Section of the potential for the $SU(2) \times SU(2)$ orbifold field theory. The diagonal line is the region where bosonic and fermionic potentials cancel one another. Global minima of negative energy lie at the points marked by triangles pointing downwards. Global maxima of positive energy are also indicated by triangles pointing upwards.*

The properties of this potential are similar to those of the orientifold model studied in the previous subsection. Conical minima are again distributed in an intrincate *landscape* with energy spacings of $O(N)$ but reaching energies of $O(N^2)$. Conical zeros of the bosonic potentials are given by flat connections of the form

$$\vec{C}_{1,2}^{(0)} = 4\pi \sum_l \vec{n}_{1,2,l}\, \nu^l \;.$$

At these points, the fermionic contribution has the value

$$V_{\rm F}(\vec{C}_1^{(0)}, \vec{C}_2^{(0)}) = -N^2\, V\left(\frac{2\pi}{N}\left(\sum \vec{n}_2 - \sum \vec{n}_1\right)\right)$$

and, as before, we have negative minima with energies ranging from $O(N)$ to $O(N^2)$ as $\sum \vec{n}_2 - \sum \vec{n}_1$ ranges from $O(1)$ to $O(N)$.

The global picture is very similar to the case of the orientifold model, including the rough statistical properties of the potential. The same analysis used before also shows that the orbifold potential averages to $O(N)$ energy while the squared potential averages to $O(N^3)$.

The similarity between the orbifold and the orientifold potential is even quantitative along the "antidiagonal" of the moduli space, i.e. the hypersurface defined by $\vec{C}_+ = 0$, where $\vec{C}_\pm = \frac{1}{2}(\vec{C}_1 \pm \vec{C}_2\,)$ are the eigenvalues of the orbifold $\mathbf{Z}_2$ action over the moduli coordinates. Along this hypersurface of $\mathbf{Z}_2$-odd connections the orbifold potential is identical to the orientifold one, up to $O(N)$ corrections,

$$V_{\rm orb}(0, \vec{C}_-) = 2\, V_{\rm ori}(\vec{C}_-) + O(N) \;.$$

Notice that this argument also implies that the origin of moduli space is unstable along the antidiagonal, in view of the similar behaviour of the orientifold potential studied in the previous section. However, there are many other unstable directions, such as $\vec{C}_1 = 0$, $\vec{C}_2 = \nu^1 \vec{c}$, for instance.

The orbifold potential also features negative-energy minima with $O(N^2)$ energy that are not inherited from down-shifting of the bosonic conical zeros. In order to exhibit these special minima, let us focus on one of the "outer rims" of the moduli space, $\vec{C}_2 = 0$, $\vec{C}_1 = \vec{C}$, and points of the form

$$\vec{C}_\epsilon = \left(2\pi \sum_{l=1}^{N-1} \epsilon_l \, \nu^l, 0, 0 \right)$$

where $\epsilon_l = \pm 1$, with $O(N/2)$ terms of each sign. The fermionic potential at these points is

$$V_{\rm F}(\vec{C}_\epsilon) = -2N \sum_i V(\nu^i \cdot \vec{C}_\epsilon\,) \sim -2N \sum_i V(\epsilon_i \pi) \sim -2N^2 \, V(\pi) \;,$$

whereas the bosonic contribution is at most of $O(N)$. To see this, notice that

$$2\pi \sum_{l=1}^{N-1} \epsilon_l \, \nu^l \cdot (\nu^i - \nu^j\,) = \pi(\epsilon_i - \epsilon_j)$$

if $i, j = 1, \dots, N-1$. All these terms are of the form $V(\pm 2\pi)$ or $V(0)$ and thus vanish. The remaining $N-1$ terms have argument

$$2\pi \sum_{l=1}^{N-1} \epsilon_l \, \nu^l \cdot (\nu^i - \nu^N) = \pi \, \epsilon_i$$

and contribute $\sum_i V(\pm\pi) \sim N\, V(\pi)$ to the bosonic potential. We conclude that these minima come from the superposition of N^2 inverted maxima of the $SU(2)$ potential.

A peculiar feature of the orbifold potential is its exact cancellation along the diagonal $\vec{C}_1 = \vec{C}_2$. This is the closest we come to the complete cancellation of the potential over the moduli space, the hallmark of the supersymmetric models. In this case, however, local minima of the potential lie outside the diagonal on the "twisted sector" of the moduli space. Although in finite volume one cannot strictly talk about spontaneous breaking of the orbifold $\mathbf{Z}_2$ symmetry, minimum energy wave functions are localized near regions with non-zero values of the "twisted fields" $\vec{C}_1 - \vec{C}_2$. In this respect, the status of this result is similar to that in [7], this time in the regime of full spatial compactification.

5. Discussion

In this paper we have examined the finite-volume dynamics of the basic examples of orbifold and orientifold models that exhibit the property of planar equivalence. We have applied standard methods based on the Born–Oppenheimer approximation to the dynamics of asymptotically-free gauge theories on small three-tori. In particular, we have analized the large-N properties of one-loop effective potentials over the moduli space of flat connections. This type of analysis leads to the known properties and microscopic determinations of the Witten index in the case of supersymmetric theories. Since planar equivalence consists precisely in a large-N "inheritance" of certain supersymmetric properties between parent and daughter theories, it becomes an interesting question whether a "planar" version of a supersymmetric index could be defined for these theories.

Examining the graded partition function as a natural candidate for such a "planar index", we have argued that the constraints imposed by perturbative planar equivalence are not strong enough to justify a useful notion of planar index, at least in the sense that it should be determined by $O(N)$ fermionic zero modes for a rank N gauge theory. Instead, we find that effective potentials over the moduli space of flat connections remain generically too large as $N \to \infty$ to allow the kind of zero-mode dynamics that is characteristic of supersymmetric theories. In particular, potential barriers of $O(N^2)$ height remain at special points in the moduli space, resulting in strong violations the much weaker property of minimal planar equivalence (cancellation of all $O(N^2)$ features). In general, planar equivalence is only maintained in a statistical sense, at the level of averages over the moduli space.

The associated wave functions show exponential localization around local minima, unlike the constant wave functions that ensure an index determined by fermion zero modes in the supersymmetric case. These large potential barriers also imply that semiclassically calculable effects on these potentials will be typically suppresed exponentially in the large-N limit. Identifying a semiclassical tunneling effect of $O(1)$ in the large-N scaling, but non-perturbative suppression in the 't Hooft coupling, would require potential barriers of order $\Delta V_{\text{eff}} \sim 1/LN$, much smaller than those found in our examples.

The orbifold model admits a very strong projection under which it behaves exactly like a supersymmetric theory. This is the diagonal projection, onto wave functions that are supported only over the diagonal of the moduli space $\vec{C}_1 = \vec{C}_2$. This projection, if exercised over the complete Hilbert space of the orbifold theory, truncates it to a supersymmetric $SU(N)$ gauge theory. Notice that the parent theory in this case is $SU(2N)$, so that we are

considering a much stronger projection here. What we find is that, to one-loop accuracy but exactly over the moduli space of flat connections, this projection can be done only over the zero-modes with identical result.

Although this behaviour of the orbifold model is interesting, it is fair to say that the required diagonal projection is completely *ad hoc* and is not supported by the dynamics of the system, which tends to favour wave functions supported on the "twisted" regions of the moduli space. In fact, our analysis agrees with previous indications from [7], suggesting that the $\mathbf{Z}_2$ symmetry might be spontaneusly broken in infinite volume, a situation that would rule out the planar equivalence realized in orbifold-type models.

It would be interesting to complement our analysis with the study of more general boundary conditions, allowing the presence of magnetic fluxes through the torus. In the case of the supersymmetric theories, this introduces interesting refinements of the index that probe non-trivial properties such as confinement and chiral symmetry breaking. In the context of the models considered in this paper, the study of such twisted sectors could reveal a better large-N vacuum behaviour in the orbifold case, and perhaps a hint of the fermion condensates that should develop in the infinite-volume theory in the case of the orientifold model. Work in this direction is in progress (c.f. [19]).

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